

The McDougal Cave and Counting issues

Sayandeep Khan, Synergy Moon LLC

March 8, 2013

Abstract

In this paper I investigate the problem of tagging elements of a set, and the elements of those elements, uniquely, when they admit an order, and two boundary elements are tagged. A heuristic sorting algorithm is also investigated.

1 Introduction

1.1 The initial problem

I will begin by investigating an imaginary problem, and then show the use of the solution in practical issues. The problem goes as follows.

Problem 1.1. Tom Sawyer is in the McDougal Cave. The cave is a system of interconnected tunnels, and rooms. Each room is connected to each other via tunnels, and inside every room, is a self-contained tunnel system. Each tunnel, may contain smaller rooms, and mazes. Tom wants to systematically inspect the cave system to find his treasure, however, he need not inspect every chamber, every tunnel and every room, and similarly, he can choose in which order he would explore. Nonetheless, he can be in only one position at a time. In order to mark which room he is in, with respect to which tunnel, and in which larger room, in turn, the tunnel itself is, Tom needs to assign a number to his location. How could such a numbering scheme work, is the question I would attempt to answer.

We will see, that this problem is similar to some issues encountered in modeling and simulation, especially, those which use blocks (assigned with certain tasks, selected from a pallet) connected to each other. I will also present yet another solution of the classic sorting problem.

1.2 Conway's Numbers - primary decisions

Conway presented a number system [1] which can be used for arbitrary counting. This system is the obvious choice for the counting problem is because it permits assignment of new tags between any two, and the construction itself admits an ordered structure which can be extended to infinity.

However, this number system does not permit too much freedom of tagging, then only one number can be created between two, and nor is this number unique to the generation, nor is the generation independent to order of generation. Moreover, it is considered that such numbers already exists. Therefore, we consider a different counting system.

2 Beyond Conway's Numbers

In this section, I propose an extension of the usual Conway's Numbers, and investigate the algebraic structure of them. Numbers generated this way will be called, in this context, "Extended Conway's Number"-s, or "ECN"-s

2.1 Axioms

Before further discussion, it is necessary to declare a few axioms.

Axiom 1. *The Axiom of Sets: Sets exist, and any set, including the null set has a few attributes, such as size etc. Sets are collection of mathematical objects, and are objects themselves.*

Sets can be assigned with natural attributes, such as size, disjoint element count, as well as synthetic attributes, e.g. position in a sequence, morphism, etc. These attributes are used for ordering sets.

Axiom 2. *The Axiom of ordering: A collection of Sets can be ordered. That is, comparative relations between sets and their properties exist. The number of possible ordering is not restricted. It is possible, two or more orderings can coincide (i.e. at this point, the requirement for unique ordering is omitted). If two or more orderings coincide, then they are equivalent, and existence of one implies the existence of the other.*

Two orderings, $a < b < \dots < c$ and $d < e < \dots < f$ are equivalent iff $a \equiv d, b \equiv e, \dots, c \equiv f$ etc. At this moment, the notion of index is not yet defined. Here the symbol \equiv implies "the same as" (dasselbe, exactly the same, of course the exactness of similarity depends on the context).

There exists the following order relations:

1. \equiv : exactly the same
2. $=$: equivalent. less strict similarity than previous
3. $<$: not (necessarily) similar
4. $>$: not (necessarily) similar, antisymmetric to the previous
5. \leq : not $>$
6. \geq : not $<$

The third and fourth relations are transformable to each other, so are the fifth and sixth.

Axiom 3. *Axiom of Numbers: Numbers are symbolic representation of properties of sets, and they exist, and can be constructed from sets. Each set has at least one number associated with it.*

If the same number can be constructed from two or more sets, then they are equivalent in that context, and vice versa. A collection of Number can form a set. Numbers can be constructed from other numbers.

Axiom 4. *Axiom of Construction: A number can be constructed from a set in different ways using well behaved functions. A number N , constructed from set S , using construction method \mathfrak{C} , can be written as $N = \mathfrak{C}(S)$.*

This axiom forms the basis of the rest of the analysis, where we attempt to construct numbers, from properties of sets, which assign an order to them.

Axiom 5. *Axiom of Repetition: Any operation can be applied on any object, an unlimited number of times, but the outcome may be undefined.*

This axiom requires immediately the following axiom:

Axiom 6. *Axiom of Operation: Set operations, including but not limited to Union, Subtraction, Intersection etc. exists.*

The behavior of these operations are defined in the Euclidean axioms.

Axiom 7. *The Euclidean Axioms apply.*

The Euclidean axioms deal with the “largeness” of an object. Largeness can have different meanings on different contexts. However, axiom of sets assume that “size”, in its intuitive sense, is a natural property of a set.

2.2 Ordering and Counting

Ordering is a natural process, therefore, the analysis can start from ordered sequence of sets.

2.2.1 Ordered Sequence of Sets and Comparison to Set

From Axiom of Ordering, sets assume ordering. Such an ordering constructs a sequence. The elements of a sequence are disjoint. The construction of a sequence does not depend on the operations described on the axiom of operations.

A sequence does not admit a size, it admits a count. A count will be defined in due course. An uniquely defined sequence is always ordered, and assumes an “ordering rule”. An ordering rule can be an enumeration, or, alternatively an (well behaved) function.

Definition 2.1. A collection of objects, brought together without those operations, which is not a sequence, is a *disarray*, or an *unordnung*. An *unordnung*, can be converted to a sequence, with a function (which is an ordering rule). Such functions are called *ordering functions*, or *ordnungsregel-s*

Two ordered sequences are equivalent, *iff* they are constructed from two disarrays, that contain pairwise equivalent elements, and same *ordnungsregel* apply on them.

A set, however is constructed using set operations.

Definition 2.2. If a set is to be constructed from elements which admit the $=$ relation under certain contexts, yet are distinct elements in a sequence, then they are handled as distinct elements under set operations. This condition is called *otherness* condition or *alteritacet* condition.

2.2.2 Order Label

Since each of the ordered sets has a number associated to it, from Axiom of Numbers, those numbers are also therefore ordered. The same ordering, therefore can be achieved by ordering the numbers associated with the sets too. Therefore:

Lemma 2.1. *Numbers are Order Labels of sets. They generate an unique ordering.*

That is, numbers label sets, and those labels can determine the ordering.

Proof. The term “Order Label”, in this context, means a symbol, which can be ordered to achieve the same ordering as the ordering of sets themselves. To prove, that numbers are Order Labels, it is sufficient to show that, $\mathfrak{O}\{N = \mathfrak{C}(S)\} \equiv \mathfrak{O}\{S\}$. Here, \mathfrak{O} is an ordering, and \equiv is the equivalence relation.

Take a collection of sets S_a, \dots, S_o . Let two orderings be $S_i < S_j < \dots < S_l$, and $S_e < S_f < \dots < S_g$. The two are equivalent iff $S_i \equiv S_e, S_j \equiv S_f, S_k \equiv S_g$.

From axiom of numbers, each set can generate an unique number, and if two sets produce the same number, then they are equivalent, and vice-versa. That is, $\mathfrak{C}(S_i) = \mathfrak{C}(S_e)$, etc implies $S_i \equiv S_e, S_j \equiv S_f, S_k \equiv S_g$. Therefore the two orderings are equivalent, if $\mathfrak{C}(S_i) = \mathfrak{C}(S_e)$, etc holds.

Under that condition, $\mathfrak{C}(S_i)$ can be generated from both S_i , and S_e . Consider the order sequence $\mathfrak{C}(S_i) < \mathfrak{C}(S_j) < \dots < \mathfrak{C}(S_l)$. This sequence implies therefore both the sequences $S_i < S_j < \dots < S_l$, and $S_e < S_f < \dots < S_g$.

Therefore, if the order sequence $\mathfrak{C}(S_i) < \mathfrak{C}(S_j) < \dots < \mathfrak{C}(S_l)$ implies two sequences, then both are equivalent. In other words, all order sequences generated by a number sequence are equivalent. QED. \square

The order labels can be used to assign an index to the elements of a sequence.

2.2.3 Count Label

A sequence admits another measure, namely

Definition 2.3. The Order Label of the last element of an ordered sequence is called the count of the sequence.

The count is useful to determine the size of set that is constructed from the sequence, but this will be explicitly defined later.

A count label is used for extracting subsequences of a sequence, then the order label of the last element of the target subsequence is the count of it. In certain cases, enumerating the order labels of the each individual element is to be avoided, and is done using count labels.

A count label is used, among other things, to assign a measure to the iterations of a repetitive operation.

2.2.4 Code Sequences and Groups

Numbers are symbols.

Definition 2.4. A string of symbols is called a codeword. A codeword can be constructed from set \mathfrak{S} of symbols, taking the elements of the set, possibly repeated, and composing them together. The complete set of such words is called language.

Usually, the symbol Σ is used instead of \mathfrak{S} , but here to avoid confusion with the summation sign, we use the symbol \mathfrak{S} .

Definition 2.5. The construction of a code sequence is given by a formal *grammar*, a set of rules, which transfers one word to another.

If the words are given, and a grammar exists that transfers two words to a third, then we arrive at the notion of Groups, and similar structures. Such structures will tell as important details of the set of numbers under investigation.

2.3 Consistence to Euclidean Axioms

The above discussion defines numbers as (symbols to) labels. However, with labels, operations can not be defined. It is therefore, necessary to unify the labels with size. Immediately it is clear that size offers a natural way of ordering sets, with the previous element of the ordered sequence being of smaller size than the next one.

2.3.1 Zahlenaufbau

Before the discussion may proceed further, certain more analysis is to be considered. This analysis will be called Zahlenaufbau in this context.

Consider the fifth axiom of Euclid: any object is larger than any part of itself, or, any part of an object is smaller than the whole. An important corollary is:

Corollary 2.1. *Since the null set is a subset (“part”) of any set, it is the smallest set.*

Axiomatically, it is proposed that any part of any object is included in the whole. So, if an object \mathfrak{o}_a contains another object \mathfrak{o}_b , then we write, $\mathfrak{o}_b \subset \mathfrak{o}_a$, $\mathfrak{o}_b < \mathfrak{o}_a$. If $\mathfrak{o}_1 \subset \mathfrak{o}_2$, $\mathfrak{o}_1 \supset \mathfrak{o}_2$, then $\mathfrak{o}_1 = \mathfrak{o}_2$.

Now, from the notion of union, one can write, $\mathfrak{o}_b \cup \mathfrak{o}_c = \mathfrak{o}_a$. Consider the general sequence :

$$\mathfrak{o} \cup \mathfrak{k} = \mathfrak{o}_{\mathfrak{k}}, \mathfrak{o}_{\mathfrak{k}} \cup \mathfrak{k}' = \mathfrak{o}_{\mathfrak{k}'}, \mathfrak{o}_{\mathfrak{k}'} \cup \mathfrak{k}'' = \mathfrak{o}_{\mathfrak{k}''} \dots \text{ ad petitem. Obviously, } \mathfrak{o}_{\mathfrak{q}} \underbrace{\cup \mathfrak{k}}_{\mathfrak{t} \text{ times}} = \mathfrak{o}_{\mathfrak{s}} \implies \mathfrak{o}_{\mathfrak{q}} < \mathfrak{o}_{\mathfrak{s}}.$$

Now, let $\mathfrak{k} = \mathfrak{k}' = \mathfrak{k}'' = \mathfrak{K} \neq \phi$, $\mathfrak{o} = \phi$, the Null set. Obviously, the largeness of the Null set is an absolute minimum. The largeness, or “size” is represented by the number Zero. \mathfrak{K} is called *einheit* from German for unitness, the property of being one. The number immediately constructed after the nullset is One, w.r.t. \mathfrak{K} , the next number is Two, w.r.t. \mathfrak{K} , et cetera. The size of each $\bigcup \mathfrak{k}$ is therefore is a number, by definition. The underscore indicates the sequence is of finite length.

If a number does not have a name, one can call it (N_1 union N_2), where N_k is named. Consider for example, the the number 39, which can be called Thirty union Nine. Notice, that here one is not dealing with positional value of a digit of a representation system.

Definition 2.6. *Natural count* is defined in the context of constant *einheit*. Consider the minimal set, which is the null set. If the elements are ordered, call the count zero. Increment this set, by repeating the union operation, to $\mathfrak{s} = \mathfrak{o} \cup \mathfrak{k} \cup \mathfrak{k} \cup \mathfrak{k} \cup \dots \cup \mathfrak{k}$. The set of the all \mathfrak{k} -s has the size of N , (in specific cases, same as the size of the resultant set \mathfrak{s}). If elements of this set is labeled and ordered, such that the count is N , then it is called a natural count. The numbers constructed, $\forall N$, thereby are natural numbers.

Remark 2.1. In this definition, the size of a set is identified with the Natural count of the *einheit* -s that construct the set. Natural counts are always defined w.r.t. an *einheit*. A natural count n with *einheit* u implies the possible existence of a set, which is constructed from $\bigcup u$, and the set of all u -s used for this construction has size n . Any value, while assigned an *einheit*, can become a natural count. The *einheit* of a natural count may be replaced with another, using certain operations.

Definition 2.7. An *einheit* is a constant set, used in repeated union to construct other sets. A *natural einheit* is the set which has the size one. Here the “one” identifies to the standard, everyday meaning of the word “one”, as in one llama, or one submarine or one Tom Sawyer.

2.3.2 Variable Einheit

Of course the choice of *einheit* is arbitrary. This leads to the following remarks:

Remark 2.2. The same number can be generated by different combinations of different einheit.

Proof. Since it is a claim of a possibility, the proof can be simply illustrated by an example. Consider the number $N = \mathfrak{k} \cup \mathfrak{k} \cup \dots \cup \mathfrak{k}$. Let $\mathfrak{k} = \mathfrak{k}_x \cup \mathfrak{k}_x$. Hence $N = (\mathfrak{k}_x \cup \mathfrak{k}_x) \cup (\mathfrak{k}_x \cup \mathfrak{k}_x) \cup \dots \cup (\mathfrak{k}_x \cup \mathfrak{k}_x)$. This is a different combination, that represents the same N . Notice that there is no boundary on the sequence length that constructs N . QED. \square

Corollary 2.2. *All numbers can be represented via different combinations of different einheits.*

Remark 2.3. One number can be a combination of unlimitedly many different einheits.

Proof. From the axioms, union between any sets is possible. Hence, any sets of any einheits exist. QED. \square

Remark 2.4. There total set of numbers is unlimitedly large.

Proof. $\forall \bigcup \mathfrak{K} = \mathfrak{o}_{\mathfrak{k}}, \exists \mathfrak{o}_{\mathfrak{k}} \cup \mathfrak{k} | \mathfrak{k} \neq \phi$. Hence, it is possible to create unlimitedly many numbers. QED. \square

Remark 2.5. There exists unlimitedly many numbers between any two given numbers.

Proof. Assume the numbers are \mathfrak{o} and $\mathfrak{o} \cup \mathfrak{K}$. $\forall \mathfrak{K}, \exists \mathfrak{K}' \subset \mathfrak{K}$.

Therefore, $\forall \mathfrak{o}, \mathfrak{o} \cup \mathfrak{K} > \mathfrak{o} \cup \mathfrak{K}' > \mathfrak{o}$, and similarly, there exists a number between $\mathfrak{o} \cup \mathfrak{K}'$ and \mathfrak{o} , and also between $\mathfrak{o} \cup \mathfrak{K}$ and $\mathfrak{o} \cup \mathfrak{K}'$, and ad infinitum. QED. \square

Remark 2.6. The set of all numbers is closed.

Proof. Any number, and any set, in this system is dealt as a set. That is, what in other system is a number, in this system is a set. Singleton elements automatically assume the status of a set. An operation on a set produces, therefore another set. Since any set can be represented in this system as a combination of different einheits, any set is a valid member of the class this system represents. Therefore, the set of all numbers (or the class of all sets) is closed, under any operation. QED. \square

2.3.3 Number Line, Number Plane and Others

The number line is an ordered sequence of all numbers generated by all einheits, that are comparable to each other. Obviously, since each einheit can generate infinite numbers, one line can contain only (strictly) two types of einheits, because, it can be expanded to infinity in two directions.

The next pair of comparable einheit can be placed along a different line, independent to the all previous lines. Hence pairwise, the einheits define a set of basis, which can be used to describe a N -dimensional space.

Example 2.1. The Positive and Negative Reals are comparable, under certain conditions, and form one number line. Positive and Negative Imaginary numbers are also under similar conditions comparable to each other, but not comparable to Reals. Therefore, the real line and the imaginary line define a number plane. This is the Argand Plane.

The comparison between any two points in a general N -dimensional space is therefore not simple. There exist N pairs of coordinates. An order relation may, however, be constructed from a particular pair. Therefore, the order relation is not unique.

An unknown number, is therefore a point in the space of unknown size. The size of real numbers are therefore real size, and the imaginaries have imaginary sizes.

2.4 Operations

The set of ECN is a formal field, with operations defined on it. We primarily define two order preserving operations, namely addition and multiplication.

2.4.1 Order Similar

Consider an order sequence $a > b > c > \dots > d$, and an operation f , such that $f(a, w) = p$, $f(b, w) = q$, $f(c, w) = r$, $f(d, w) = s$, for some w . If $p < q < r < \dots < s$ or $p > q > r > \dots > s$, then call the function an order similar one. Given the sequence $a > b > c > \dots > d$, if $p > q > r \dots$ etc, then it is a monotonic operation, otherwise it is an antitonic operation.

Remark 2.7. On a given set, a monotonic operation and an antitonic operation is isomorphic. An order similar operation implies its inverse is also order similar. The proof is trivial.

We investigate the invariant(s) of order similar operations. It is trivially clear that

Remark 2.8. The closure, and the initial (resp. final) segments are not necessarily invariants to a order similar operation. They are invariants for a monotonic operation. In order theory, the monotonic operation is known as order preserving.

However, isomorphism implies, that the range and image of the operation has the same natural count of elements. Formally:

Remark 2.9. The height, defined as the natural count of the sequence (resp. subsequence) – einheit, of the sequence (resp. subsequence) is an invariant, under order similar operations.

Other invariants may be found from the definition of order similar operations. Here we consider the concept of neighborhood, and introduce the concept of *zwischenraum*.

Consider a sequence $S = a, b, c, \dots, d$, such that, $a < b < c < \dots < d$ apply. Assign order labels $\alpha, \beta, \gamma, \dots, \delta$ on them respectively to represent them using a graph. A subsequence s of S is a, b, c . If we represent s with a graph g , then it is trivially clear that the neighbors of b are a and c . Similar statements apply for every subsequence of length 3 in S . Under an order similar operation, f , $f(b)$ has the neighbors (in a graph-representation with graph g') $f(a)$ and $f(c)$. From remark 3.13, it is trivially clear that g and g' are isomorphic, specifically, b and $f(b)$ are. Hence they have isomorphic neighborhood. Same applies for any subsequence in S . Hence the neighborhood matrix is invariant.

Definition 2.8. This permits to define a property of such 3-tuples, which we will call *Zwischenraum* (from German “space in-between”) invariance. Consider a sequence $S : a, b, c, d, \dots, z$. Let $a < b < c$, such that, $\nexists d | a < d < b \vee b < d < c$. Then, b is said to occupy the *Zwischenraum* of a and c , symbolically, $b \in 3(a, c)$. Clearly, $f(b)$ is occupying the *zwischenraum* of $f(a)$ and $f(c)$, if f is an order preserving operation. Therefore, the *zwischenraum* for each 2-tuple, constructed from an ordered sequence, is uniquely defined by a single element e , the *zwischenraum* of the image of the 2-tuple is uniquely defined with the image of e . This property is called *Zwischenraum Invariance*.

We try to generalize this for cases, where the *zwischenraum* of two elements will consist of an ordered set s consisting of more than one elements. The *zwischenraum* of the image of the two bounding elements consist of the image of s . This is the sought generalization. Moreover, the set s itself is also subject to *zwischenraum* invariance, and so is any subset of s . Formally:

Theorem 2.1. Any sequence with natural count larger than 2, if admits a *zwischenraum* invariance, then any subsequence of it of length having natural count 2 admits *zwischenraum* invariance to the same operation.

2.4.2 Addition and Multiplication

Addition Addition is a monotonic binary operation, symbolized with $+$, with following properties.

- If $A < B < C < \dots < D$, then $A + k < B + k < C + k < \dots < D + k$, for all A, B, C, \dots, D, k
- $(p + k) \setminus p$ does not depend on p , but on k , for all p, k
- $a + p$ is either $< a$ or $> a$, both may hold. This is applicable for all a, p
- Axiom of associativity and axiom of commutativity holds.

Lemma 2.2. *The addition operation may be represented by set union.*

Immediately, one realizes, that there exists at least one p , for which $a + p = a$. Such p -s are called neutral to addition. The addition operation, from Euclidean axioms, is comparable to set unions. Let u be the einheit. $a + p$ is defined as the size of $a \cup \bigcup u$, where the count of $\bigcup u$ is p .

Multiplication Multiplication, an order similar operation, is similarly defined, symbolized with \times or $*$, where

- If $A < B < C < \dots < D$, then either $A * k < B * k < C * k < \dots < D * k$, or $A * k > B * k > C * k > \dots > D * k$ for all A, B, C, \dots, D, k
- $(p * k) \setminus p$ depends on both p , and k , for all p, k
- $(a * p)$ is either $< a$ or $> a$, both may hold. This is applicable for all a, p
- Axiom of associativity and axiom of commutativity and axiom of distributivity holds.

There also exists neutral to multiplication, such that $a * p = a$. Further, $a * p$ is the size of $\bigcup a$, where, count of $\bigcup a$ is p .

Both operations will be extended in forthcoming discussion to account for different types of einheit.

Claim 2.1. The total set of all numbers, written \mathbf{n} , is a group.

Proof. \mathbf{n} is a set, with two operations defined on it, and is closed, by definition, and from remark 3.11 to both. Both operations agree to the axiom of associativity, and assume an identity element. In next sections, the inverses will be discussed. Hence group axioms are fulfilled. QED. \square

2.4.3 Operands of $+$ and \times

Above the operations are described in terms of Order Labels, which are sizes of sets. The size of sets are identified with the natural count of einheits that construct the set.

Sets need not be collection of disjoint elements, the elements may touch (in a topological sense) or overlap each other. In practicality, it is more often the case than not, that, the elements are not disjoint.

The addition operation deals with the size of the sets, which is intuitively clear, and is supported by Euclidean axioms. If both operands possess the same einheit, then the result also has the same einheit as well. Otherwise, either one einheit is transformed (we will use the term transcribed, i.e. represented differently, instead) to the other, and then the result is computed, or they are left as disjoint union.

Example 2.2. Assume the addition between a and b is performed (which is the union of two underlying sets having natural counts a and b), where a is imaginary, and the other is real (the imaginaries are not yet defined explicitly). Recall each natural count is defined w.r.t. a constant *einheit*. In this particular case the sum is left as $a + b$, because the real and the imaginary *einheit* is separate. Had they both the same *einheit*, then the result would be represented by c , which is the natural count of the set $A \cup B$, A and B having natural counts a and b respectively. The natural count c will be defined w.r.t. the same *einheit* as both a and b .

The multiplication operation is however, slightly more complex. If the operation is performed on two operands, with two (not necessarily same) *einheits*, for example to calculate $a \times b$, a set A with size a is constructed, w.r.t the *einheit* of a . b is now defined as a natural count, w.r.t. its own *einheit*. A sequence S , each of its element being a , is constructed with count b . The count labeling need not be dependent on the size. Another set R is constructed by repeated union of elements of S . Then R is the result of the multiplication operation.

2.4.4 Further Operations on \mathbf{n}

Although the purpose of \mathbf{n} here is to serve simply as a tagging system, that permits somewhat flexible re-tagging system, and possibility to generate new numbers between any two in a comprehensive way. It is now of interest, to investigate, if \mathbf{n} assumes an algebraic structure that can be exploited.

Length of a Line Segment Given two points \mathbf{p}_1 and \mathbf{p}_2 in the number line \mathbf{l} , where $\mathbf{p}_1 < \mathbf{p}_2$ holds, assume there exists a third number \mathbf{p}_3 | $\mathbf{p}_1 + \mathbf{p}_3 = \mathbf{p}_2$, where the $+$ operation is defined above. Since \mathbf{p}_3 is also a number, it exists in \mathbf{n} . \mathbf{p}_3 is called the length of line segment between \mathbf{p}_1 and \mathbf{p}_2 .

Now, to calculate \mathbf{p}_3 , we need $\sim \mathbf{p}_1$, the inverse of \mathbf{p}_1 w.r.t addition with \mathbf{p}_2 . We have $\mathbf{p}_1 + \mathbf{p}_3 = \mathbf{p}_2$. From previous discussion, the operands are sizes, hence the Euclidean axioms apply. Hence we add $\sim \mathbf{p}_1$ to both sides. RHS is the \mathbf{p}_3 + neutral to addition, which is just \mathbf{p}_3 and LHS is $\mathbf{p}_2 + (\sim \mathbf{p}_1)$.

This discussion leads to the length operation. The length operation is written as $|a - b|$. It is defined as $a + (\sim b)$, *iff* $a < b$, $b + (\sim a)$ *otherwise*.

Remark 2.10. The outcome of this operation has the same *einheit* as the operands.

The definition of a subtraction operation follows immediately. The Euclidean axioms use the notion of “taking away”, which is comparable to set exclusion operation. Translated to notion of order labels, $a - b$, where the $-$ symbol designates the subtraction.

Remark 2.11. From Euclidean axioms, subtraction is only then defined, when $a \succ b$.

$a - b$ therefore is the size of the set $A \setminus B$, where A has the size a , and B the size b . If the underlying sets have similar *einheit*, then the result is also the same unity.

For any a , it is immediately clear, that $a - a = 0$, the size of the null set. The 0 is again the neutral to addition. So, $a - a = 0 = a + \smile a$. Now, again, from Euclidean axioms, adding any b to both sides, preserve the equality relation. Therefore, the subtraction operation is same as adding the inverse.

Negative Numbers and Extension A negative numbers may be treated as the inverse to a given number in a group under addition. The definition is rather synthetic, that it does not immediately follow from Euclidean axioms. Inverses of negatives $((a^{-1})^{-1} = a$ holds) are called positives.

Claim 2.2. Each set has an inverse of it, such that set $+$ inverse of it $= \phi$. Those inverses follow the Euclidean axiom. These inverses may be just mathematical concepts.

Proposition 2.1. *In this context, we define the the order relation as follows. Let $\smile a$ and $\smile b$ be two negative numbers. $\smile a > \smile b$ iff $a > b$.*

Definition 2.9. Call the numbers with natural einheits *Universal Positives*, and the corresponding negatives *Universal Negatives*.

Remark 2.12. In particular einheit scheme, negatives has the same einheit as the corresponding positives, then they are inverses to each other, and results in a neutral, under addition.

Proposition 2.2. *Sometimes, it is necessary to order the positives and the negatives (or other pairs) in the same sequence. For that purpose, the number line is constructed.*

Universally, assign $\smile a < a$. Start $\smile 0$, and continue till $\smile \infty$, then the sequence is $\smile 0 < 0 < \dots < \smile a < a < \dots < \smile \infty < \infty$. This ordering permits a semi infinite line to be constructed. If, however, $\smile a > \smile b$ iff $a < b$ is used, together with $\smile a < 0$ if $\smile a$ is a universal negative, then we arrive at the usual positive-negative ordering.

Division as Multiplication with Inverses Division operation is defined for multiplication. If $a \times b =$ neutral of multiplication, then, $c \div a = c \times b$. Division operation is therefore also order similar.

The division operation splits the set c in parts, the set of which admits a count. The count is a .

2.4.5 The Algebraic Structure of \mathfrak{n}

Quasigroups of \mathfrak{n}

Theorem 2.2. *A subset $\mathfrak{s} \subset \mathfrak{n}$ that contains all numbers of same einheit, and only numbers of same einheit is a group, under the addition operation.*

Proof. Assume $a, b \in \mathfrak{s}$. The addition operation results in c . c has the same einheit as a and b . Therefore, $c \in \mathfrak{s}$. this holds $\forall a, b \in \mathfrak{s}$. Therefore $a, b \in \mathfrak{s} \implies a + b \in \mathfrak{s}$. Hence \mathfrak{s} is a magma.

$\forall a, c$, we have $b = c - a$. Note that although the subtraction operation requires the presence of an inverse, we do not require the inverse to be present in \mathfrak{s} itself. \mathfrak{s} is therefore a quasigroup.

0 is the element which is the neutral to addition, and $0 \in \mathfrak{s}$. Therefore \mathfrak{s} is a loop.

Associativity of addition holds. Hence \mathfrak{s} is a group. \square

This theorem can be trivially extended to :

Theorem 2.3. *A subset $\mathfrak{s} \in \mathfrak{n}$ which contains all numbers of a particular einheit, and all it's inverses, and only the mentioned numbers, is a group.*

Then we consider the segment length operation. The length segment operation in general is not associative, nor does it admit an unique definition of division. Therefore it does not admit a group-like structure.

Construction of a Finite Group Given $a, b, c \in \mathfrak{s} \subset \mathfrak{n}$, we consider if \mathfrak{s} can be constructed in a group, based on the order properties. The addition operation automatically admits a group structure.

The multiplication operation depends on the assignment of count. It can be trivially shown that :

Theorem 2.4. *If the subset $\mathfrak{s} \subset \mathfrak{n}$ contains only positives, then it admits a group structure. If the subset contains only negatives, then assigning negative counts to multiplication admits a group structure. If $\mathfrak{s} \subset \mathfrak{n}$ contains both positives and negatives, then under the usual multiplication scheme, it admits a group structure. Same applies for division operation.*

Now consider the transformation group containing the operations defined above. Trivial investigation shows:

Theorem 2.5. *The sets of operations addition, multiplication (division), respectively, and count assignment all admit a group structure.*

Obviously each of them are infinite groups. The composition is simply the convolution of two operations. Such groups are extremely simple in structure, and of less interest in this context.

We next consider groups whose elements take the form $\backslash \mathfrak{a}$, where \backslash is a operation, and a is a number. If \backslash is the addition operation, then the structures are similar to what is seen with the addition group already.

However, interesting observations can be made if $\backslash : \times$. This groups contains a subgroup, which is constructed of non-prime multiples of natural einheits and their inverses. This subgroup is also a normal subgroup. A series of quotient groups to this is immediately generated. Each of them is also abelian.

These groups however does not admit a soluable structure. Nonetheless, we consider the order properties.

Theorem 2.6. *Each such normal subgroup $\mathfrak{p} \triangleleft \mathfrak{n}$ is \mathfrak{n} orderable, then they admit an order which is invariant under $n\mathfrak{p}n^{-1} \forall n \in \mathfrak{n}$*

The proof is trivial. Further investigation shows \mathfrak{p} does not contain any subgroup.

2.5 Generation of Numbers and Different Einheits

Armed with the definitions of numbers and usual operations, the next section is dedicated to the original question.

2.5.1 Generation

Assuming two non-neighboring elements are tagged, of extreme interest is to find a method to tag the elements in between the tagged ones. For that, the set inbetween the two tagged elements is split, at will, and split sequence is assigned with a count.

Therefore, there are two steps inherent to the process. The splitting operation and the count assignment operation, from theorem 3.5 assumes a structure of a finite group.

Formally, assume a chain $\mathfrak{a}_1 \in \mathfrak{a}_2 \in \mathfrak{a}_3 \in \dots \in \mathfrak{a}_n$. Assume further that $\mathfrak{b}_1, \mathfrak{b}_2 \in \mathfrak{a}_n$ are tagged, and they are NOT neighboring elements. Then, we want to tag all elements in \mathfrak{a}_1 .

Trivially, and using the discussions above, it is clear that sets may be tagged independent to each other.

Uniqueness Requirements We consider the uniqueness of each tagging. Clearly, each \mathfrak{a}_i has a different einheit. However, the tagging of objects between $\mathfrak{b}_a, \mathfrak{b}_b \in \mathfrak{a}_i$ is definitely different than $\mathfrak{b}_a, \mathfrak{b}_b \in \mathfrak{a}_j$. Therefore $\mathfrak{b}_a, \mathfrak{b}_b \in \mathfrak{a}_i$ must be different numbers than $\mathfrak{b}_a, \mathfrak{b}_b \in \mathfrak{a}_j$. This difference is created by different einheits of \mathfrak{a}_i and \mathfrak{a}_j .

Assume $\mathfrak{b}, \mathfrak{c} \in \mathfrak{a}_i$ are tagged. \mathfrak{a}_i has einheit u_i . Splitting $\mathfrak{b} - \mathfrak{c}$ results in einheit u_j . Then $\mathfrak{d} | \mathfrak{b} < \mathfrak{d} < \mathfrak{c}$ has einheit $u_i + u_j$.

2.5.2 Construction of Finite Group and Consequences

We consider the division (splitting) operations that take the form $\times \frac{1}{p}$, where p is a number with a natural, positive, einheit.

It has been shown in the previous discussion that if p is not a prime number, then the set of the said operations form a normal subgroup, and there exists quotient groups to this.

Therefore, the tagging algorithm can take the advantage of creating a quotient group, which still preserves the ordering, the einheit, yet accommodates for newer symbols than the original group. That is, the measure of $\mathfrak{b} - \mathfrak{c}$ is changed, and is a natural multiple of the original value. If the original tagging had a natural count, is not however preserved. Therefore, a new natural count can be

assigned with elements, making place inserting new symbols. One can trivially show that:

Theorem 2.7. *(An operation from) The quotient group preserves the order. If a triple is considered, with the initial elements that serve as the boundary, and the set whose elements are to be ordered in between, then this triple is also preserved.*

Since there exists normal subgroups, and the elements \mathfrak{a}_i -s of the subset chain can be accessed independently to each other, the splitting operation, as the tagging also, can be done independent to each other. The result can be uniquely determined if the respective einheit is known.

Ordering of numbers having composite einheit in form $u_1 + u_2 + \dots + u_n$ is done as follows. Rewrite the two comparable numbers as $\mathfrak{s}_1 u_1 + \mathfrak{s}_2 u_2 + \dots \mathfrak{s}_n u_n$, such that $\mathfrak{v} : u_1 > u_2 > \dots > u_n$ holds. Then let u_i be the first einheit in \mathfrak{v} , where the corresponding \mathfrak{s}_i is different in the two numbers. Order the \mathfrak{s}_1 , that is the order of the two numbers

3 Solution to the Original Problem and Application

3.1 Solution of the McDougal Problem

Now, the McDougal cave problem admits a trivial solution. Attach a different einheit to each cave and chamber and tunnel. Require the einheit corresponding to one object that is included in another be lesser than that of later. If an object, inbetween two is to be tagged, while the both of them are already tagged, then tag them with a number generated from that of the former two.

The application of this system finds use in controlling the execution of any computer simulation, and in database manipulation. Indeed it is possible to tag a sequence of similar objects in McDougal cave as 21,22,23,...35... etc, but the change of the second digit from right may make the fact less obvious that the objects are actually the same. Therefore proposed tagging is $2 \oplus 1, 2 \oplus 2, \dots, 2 \oplus 267, \dots$ etc, where \oplus designates the addition of different einheits, and the tagging implies that all of them belongs to the zwischenraum between 2 and 3, (with einheit larger than the objects in consideration). If there are n such object, then the tag for the m -th one can be generated from 2, 3, $\{n, m\}$ using the method illustrated above.

If, a new object between two tagged one is to be inserted, then, the problem is a bit complex, because m is a natural count. Therefore, a re-tagging is necessary. However, a quotient group sorts the issue - for example, you can insert an object in the second place, in the sequence, $2 \oplus 1, 2 \oplus 2, 3 \oplus 1, 3 \oplus 3$, by re-tagging the first and second, to $2 \oplus 1$ and $2 \oplus 3$, where the second tagging is an operation in a quotient group of the group in which the first tagging belongs. The quotient group preserved the ordering, and the terminal tags. This greatly simplifies the process for block based heuristic simulation algorithms.

3.2 Sorting

We consider the sorting problem. Here, sorting is done heuristically, and an element from the unsorted sequence is inserted in the sorted sequence, in it's place.

The algorithm measures the rate of change of values as it progress along the sequence starting from a particular boundary. Then it predicts a location where a new, known element can be inserted. If the insertion does not disrupt the order, then insertion is followed by algorithm termination. Otherwise, taking the predicted location as a boundary, the algorithm repeats itself.

Infinity loops are not avoidable. In case of an infinity loop, the algorithm fails.

This algorithm, is works at the worst case, is of complexity $O(n)$.

References:

[1]: Gonsor, H. An introduction to the theory of Surreal Numbers, Cambridge University Press, 1986